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Three-dimensional $\mathcal{N} = 8$ conformal supergravity and its coupling to BLG M2-branes

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Abstract: This paper is concerned with the problem of coupling the $\mathcal{N} = 8$ superconformal Bagger-Lambert-Gustavsson (BLG) theory to $\mathcal{N} = 8$ conformal supergravity in three dimensions. We start by constructing the on-shell $\mathcal{N} = 8$ conformal supergravity in three dimensions consisting of a Chern-Simons type term for each of the gauge fields: the spin connection, the SO(8) R-symmetry gauge field and the spin 3/2 Rarita-Schwinger (gravitino) field. We then proceed to couple this theory to the BLG theory. The final theory should have the same physical content, i.e., degrees of freedom, as the ordinary BLG theory. We discuss briefly the properties of this "topologically gauged" BLG theory and why this theory may be useful.

Keywords: Extended Supersymmetry, Supergravity Models, M-Theory

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Contents

1 Introduction

Recently a basically unique three-dimensional maximally ($\mathcal{N} = 8$) superconformal theory was constructed by Bagger and Lambert, and by Gustavsson (BLG) [1–4]. It is the purpose of this paper to develop the corresponding ($\mathcal{N} = 8$) conformal supergravity theory and couple it to the BLG theory.

The BLG theory, containing a Chern-Simons gauge field coupled to matter fields, was originally proposed to describe multiple M2-branes. An interesting aspect of the fact that the BLG theory is a Chern-Simons theory [5] is its potential importance also in the context of condensed matter applications. The multiple M2-brane interpretation has, however, met with a number of problems related to the algebraic structure of the theory. The BLG construction is based on a four-index structure constant for a three-algebra with a Euclidean metric. This three-algebra is known [6, 7] to have basically only one realization, $\mathcal{A}_4$, related to the ordinary Lie algebra $so(4)$. This seems to be limiting the role of the BLG theory to stacks of two M2-branes [8–10].

It may be of some interest to couple the BLG theory to supergravity. In fact, in the context of AdS_5/CFT_4, similar couplings of a superconformal field theory to its supergravity counterpart have been considered in the past, see, e.g., [11][1] and references therein. A coupling to supergravity also provides a framework for curved M2 branes and may perhaps be used in a way similar to how quantum properties of the string are usually defined. (This may be more natural in the context of the ABJM $\mathcal{N} = 6$ theory [10] which can describe one as well as many M2 branes.[2] This theory can most likely be coupled to conformal supergravity in the same way as done here for the BLG theory.) The geometric description

[1] We are grateful to Arkady Tseytlin for discussions on this point.

[2] For several reasons one may, in fact, suspect that globally there is no distinction between one and several M2 branes.

of the superstring coupled to supergravity, generally referred to as the Polyakov string, was first given in [12] and later used by Polyakov [13] to define the string at the quantum level. There is, of course, an interesting issue at this stage for such an interpretation to be viable in the M2 case, related to the fact that BLG/ABJM type theories appear to be in a static gauge of a would-be covariant theory. We have no comments about this at the moment and regard this work only as a possible step in this direction.

However, for the coupling to gravity to make sense in this latter context, the coupled three-dimensional BLG (or ABJM) theory should not pick up any new propagating degrees of freedom. Thus the supergravity theory needs to be special, in some sense topological before being coupled to matter. For $\mathcal{N} = 1$ there is such a theory in three dimensions as shown by Deser and Kay in [14]. It consists of two Chern-Simons type terms, one for the spin connection and one for its superpartner the Rarita-Schwinger field. Although none of them are conventional Chern-Simons terms (e.g., the spin connection is constructed from the dreibein), we will refer to both as Chern-Simons terms. Some issues related to the physical content of theories of this kind have been addressed in [15, 16]. For instance, the equation of motion for the dreibein in the pure gravity case restricts the geometry to be conformally flat [15].

In this paper we construct the $\mathcal{N} = 8$ version of this supergravity theory which interestingly enough turns out to be rather simple; starting from the Deser-Kay $\mathcal{N} = 1$ theory [14] one just gives the spinors an extra SO(8) spinor index and adds a Chern-Simons term for the corresponding R-symmetry gauge field. It is then rather straightforward to show that this theory is invariant under the local symmetries, supersymmetry, special superconformal, and dilatations. (The Lagrangian of this theory can also be obtained by starting from the gauged superconformal algebra and subject it to curvature constraints as shown in [17, 18].) It is then possible to couple this conformal supergravity theory to the BLG theory using familiar methods. In this paper we will perform this coupling up to some higher order interaction terms between the two sectors. The resulting theory will here sometimes be referred to as the topologically gauged BLG theory since the global symmetries of the BLG theory, namely Poincaré, $\mathcal{N} = 8$ supersymmetry and SO(8) R-symmetry, are here all being gauged by the introduction of gauge fields and the corresponding Chern-Simons terms. The introduction of levels k can be done separately in the BLG sector [8–10] and in the supergravity sector [15] raising some interesting questions. This is discussed further in the last section.

The paper is organized as follows. In section two we construct the $\mathcal{N}=8$ conformal (or topological) supergravity by writing down an on-shell Lagrangian containing only three types of Chern-Simons terms, one for each gauge symmetry. We then explicitly demonstrate that this supergravity theory has the required $\mathcal{N}=8$ local symmetries. In section three we review the BLG theory and present in detail the coupling of it to the $\mathcal{N} = 8$ conformal supergravity given in section two. The last section contains conclusions and some further comments.

2 Pure topological $\mathcal{N} = 8$ supergravity in three dimensions

The off-shell field content of three-dimensional $\mathcal{N} = 8$ conformal supergravity is

$$e_\mu{}^\alpha\ [0],\ \chi_\mu^i\ \left[-\frac{1}{2}\right],\ B_\mu^{ij}\ [-1],\ b_{ijkl}\ [-1],\ \rho_{ijk}\ \left[-\frac{3}{2}\right],\ c_{ijkl}\ [-2], \tag{2.1}$$

where we have given the conformal dimension in brackets after each field. This set of fields constitute an off-shell multiplet of $\mathcal{N} = 8$ three-dimensional conformal supergravity [19] as indicated by the degree of freedom count (which is just as in four dimensions but then on-shell). The task now is to construct a topological Lagrangian from a set of Chern-Simons terms. In fact, by checking which scale invariant terms can be constructed from the above set of fields one concludes that the last three fields will satisfy algebraic field equations. This means that we can construct the on-shell Lagrangian using only the three gauge fields of 'spin' 2, 3/2 and 1, i.e. $e_\mu{}^\alpha[0]$, $\chi_\mu^i[-\frac{1}{2}]$, $B_\mu^{ij}[-1]$. (Note that the i index used here corresponds in the following to the SO(8) spinor index that is not explicitly written out for the supersymmetry parameter. The R-symmetry gauge field in the adjoint of SO(8) may, due to triality, be given a pair of antisymmetric indices in any of the three eight-dimensional representations.)

Inspired by the work of Deser and Kay [14], van Nieuwenhuizen [17], and Lindström and Roček [18], we start from a Lagrangian of the form[3]

$$\begin{aligned} L = {} & \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha\left(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho\right) - \epsilon^{\mu\nu\rho}Tr_i\left(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho\right) \\ & -ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_\sigma), \end{aligned} \tag{2.2}$$

where $\tilde{\omega}$ is the spin connection and the traces in the first and second terms are over the vector representation of the Lorentz group SO$(1,2)$ and the R-symmetry group SO(8), represented by indices α and i, respectively. Note that the coefficient in front of the R-symmetry Chern-Simons term may seem non-standard but as we will see below the $\mathcal{N} = 8$ supersymmetry properties depends crucially on the value of this coefficient.

We will frequently use the standard notation [14]

$$f^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{D}_\nu\chi_\rho, \tag{2.3}$$

which makes the Rarita-Schwinger term read

$$-4i\bar{f}^\mu\gamma_\beta\gamma_\alpha f^\nu(e_\mu{}^\alpha e_\nu{}^\beta e^{-1}), \tag{2.4}$$

where we have spelt out explicitly all dependence of the dreibein that needs to be varied when checking supersymmetry.

The standard procedure to obtain local supersymmetry is to start by adding Rarita-Schwinger terms to the dreibein-compatible ω in order to obtain a supercovariant version of it. That is

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + K_{\mu\alpha\beta}, \tag{2.5}$$

[3] The Lagrangian used here was in fact given in [18] based on a generalization of the superconformal algebra method of [17]. We will, however, base our discussion entirely on methods related to those of Deser and Kay in [14].

where

$$\omega_{\mu\alpha\beta} = \frac{1}{2}(\Omega_{\mu\alpha\beta} - \Omega_{\alpha\beta\mu} + \Omega_{\beta\mu\alpha}), \tag{2.6}$$

with

$$\Omega_{\mu\nu}{}^{\alpha} = \partial_\mu e_\nu{}^{\alpha} - \partial_\nu e_\mu{}^{\alpha}, \tag{2.7}$$

and

$$K_{\mu\alpha\beta} = -\frac{i}{2}(\bar{\chi}_\mu\gamma_\beta\chi_\alpha - \bar{\chi}_\mu\gamma_\alpha\chi_\beta - \bar{\chi}_\alpha\gamma_\mu\chi_\beta). \tag{2.8}$$

This combination of spin connection and contorsion is supercovariant, i.e., derivatives on the supersymmetry parameter cancel out if $\tilde{\omega}_{\mu\alpha\beta}$ is varied under the ordinary transformations of the dreibein and Rarita-Schwinger field:

$$\delta e_\mu{}^{\alpha} = i\bar{\epsilon}\gamma^\alpha\chi_\mu, \;\; \delta\chi_\mu = \tilde{D}_\mu\epsilon. \tag{2.9}$$

The covariant derivative appearing in the Lagrangian and in the variation of the Rarita-Schwinger field takes the following form acting on a spinor

$$\tilde{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon + \frac{1}{4}B_{\mu ij}\Gamma^{ij}\epsilon, \tag{2.10}$$

that is, both the Lorentz SO(1, 2) and the R-symmetry SO(8) groups are gauged. Note that the spinors in the gravity sector, i.e., the susy parameter and the Rarita-Schwinger field, are of the same SO(8) chirality while the spinor in the BLG theory is of opposite chirality.

Our goal now is to show that the above Lagrangian is $\mathcal{N} = 8$ supersymmetric (up to a total divergence) under the above transformations of the dreibein and the Rarita-Schwinger field together with a transformation of the SO(8) R-symmetry gauge field $B_{\mu ij}$ that will be determined in the course of the calculation. This superconformal $\mathcal{N} = 8$ supergravity theory will then be coupled to the BLG theory in the next section.

We will derive the variation of the Lagrangian following closely the steps in the $\mathcal{N} = 1$ case given by Deser and Kay in [14]. There is, however, good reason to be somewhat more explicit than in that paper since we do it for $\mathcal{N} = 8$ and will need to spell out in detail where the two calculations differ. Our derivation will make use of a Fierz basis (see the appendix) which will turn out to simplify the calculations quite a bit.

Introducing the dual SO(8) R-symmetry and curvature fields (see [14])

$$G^{*\mu}_{ij} = \frac{1}{2}\epsilon^{\mu\nu\rho}G_{\nu\rho ij}, \;\; \tilde{R}^{*\mu}{}_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{R}_{\nu\rho\alpha\beta} \tag{2.11}$$

and similarly for $\tilde{\omega}$, as well as the double and triple duals

$$\tilde{R}^{**\mu,\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\tilde{R}^{*\mu}{}_{\beta\gamma}, \;\; \tilde{R}^{***}_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\alpha}\tilde{R}^{**\nu,\alpha}, \tag{2.12}$$

where in the last expression only the contorsion part of the Riemann tensor contributes. In fact, one can show that

$$\tilde{R}^{***}_{\mu} = ie^2\bar{\chi}_\nu\gamma_\mu f^\nu. \tag{2.13}$$

From the fact that the affine connection and spin connection are related by

$$\Gamma^{\rho}_{\mu\nu} = \omega_\mu{}^{\alpha}{}_{\beta}e_\nu{}^{\beta}e_\alpha{}^{\rho} + e_\alpha{}^{\rho}\partial_\mu e_\nu{}^{\alpha}, \tag{2.14}$$

and that the variation of the affine connection is

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(D_\mu\delta g_{\nu\sigma} + D_\nu\delta g_{\mu\sigma} - D_\sigma\delta g_{\mu\nu}), \tag{2.15}$$

we find directly that

$$\delta\tilde{\omega}^{*\alpha}_{\mu} = -2i\left(\bar{\epsilon}\gamma_\mu f^\alpha - \frac{1}{2}e_\mu{}^\alpha\bar{\epsilon}\gamma_\nu f^\nu\right). \tag{2.16}$$

Combining this result with the fact that the commutator of two supercovariant derivatives, acting on a spinor, is

$$[\tilde{D}_\mu, \tilde{D}_\nu] = \frac{1}{4}\tilde{R}_{\mu\nu\alpha\beta}\gamma^{\alpha\beta} + \frac{1}{4}G_{\mu\nu ij}\Gamma^{ij}, \tag{2.17}$$

we find that the symmetric part of $R^{**\mu,\alpha}$ cancels in the supersymmetry variation of the dreibein and gravitino Chern-Simons terms. Performing also the variation of the Chern-Simons term for the SO(8) gauge field we find that also $G^{*\mu}_{ij}$ cancels provided we choose the variation of $B_{\mu ij}$ to be

$$\delta B^{ij}_{\mu} = -\frac{i}{2}e^{-1}\bar{\epsilon}\Gamma^{ij}\gamma_\nu\gamma_\mu f^\nu. \tag{2.18}$$

Inserting these variations into δL gives

$$\begin{aligned}
\delta L &= \delta L_1 + \delta L_2 + \delta L_3 + \delta L_4,\\
\delta L_1 &= 4\bar{\epsilon}(\gamma_\alpha\gamma_\beta f^\alpha)\bar{f}^\mu\gamma^\beta\chi_\mu,\\
\delta L_2 &= 8\bar{f}^\mu(\gamma_\alpha\gamma_\beta f^\alpha)\left(\bar{\epsilon}\gamma^\beta\chi_\mu - \frac{1}{2}e_\mu{}^\beta\bar{\epsilon}\gamma^\nu\chi_\nu\right),\\
\delta L_3 &= 4(\bar{f}^\alpha\gamma_\beta\gamma_\alpha)\gamma_\gamma\chi_\mu\epsilon^{\beta\mu\nu}\left(\bar{\epsilon}\gamma_\nu f^\gamma - \frac{1}{2}e_\nu{}^\gamma\bar{\epsilon}\gamma^\rho f_\rho\right),\\
\delta L_4 &= -\frac{1}{2}(\bar{f}^\alpha\gamma_\beta\gamma_\alpha)\Gamma^{ij}\chi_\mu\epsilon^{\mu\beta\gamma}\bar{\epsilon}\Gamma_{ij}(\gamma_\delta\gamma_\gamma f^\delta).
\end{aligned} \tag{2.19}$$

In order to show that the variation of the Lagrangian vanishes some of the terms in the above expression must be rearranged by Fierz transformations. As we will see later it will turn out to be convenient to review the $\mathcal{N}=1$ case before turning to the more complicated case of $\mathcal{N}=8$. To proceed in a systematic manner we have chosen to pick a basis of $\mathcal{N}=1$ expressions consisting of $(\bar{f}\ldots f)(\bar{\epsilon}\ldots\chi)$ where the dots correspond to either a charge conjugation matrix or such a matrix times a three dimensional gamma matrix (recall that all the spinors are Majorana). An independent set of such expressions is defined in the appendix.

By applying the Fierz transformations to δL_1 and δL_3 above and expressing all terms so obtained in the Fierz basis one can show, after some $\mathcal{N}=1$ Fierz calculations, that they exactly cancel δL_2. This is the result of Deser and Kay [14].

It now becomes rather easy to establish that also for $\mathcal{N}=8$ the variation will vanish when δL_4 is included and use is made of the full $\mathcal{N}=8$ Fierz identity for SO(8) spinors of the same chirality, i.e.,

$$\begin{aligned}\bar{A}B\bar{C}D = -\frac{1}{16}\Big(&\bar{A}D\bar{C}B + \bar{A}\gamma_\alpha D\bar{C}\gamma_\alpha B\\ &-\frac{1}{2}\bar{A}\Gamma^{ij}D\bar{C}\Gamma^{ij}B - \frac{1}{2}\bar{A}\gamma_\alpha\Gamma^{ij}D\bar{C}\gamma_\alpha\Gamma^{ij}B\\ &+\frac{1}{48}\bar{A}\Gamma^{ijkl}D\bar{C}\Gamma^{ijkl}B + \frac{1}{48}\bar{A}\gamma_\alpha\Gamma^{ijkl}D\bar{C}\gamma_\alpha\Gamma^{ijkl}B\Big).\end{aligned} \tag{2.20}$$

The argument is as follows. From the $\mathcal{N}=1$ case, for instance by using the basis given in the appendix and the $\mathcal{N}=1$ Fierz identity

$$\bar{A}B\bar{C}D = -\frac{1}{2}(\bar{A}D\bar{C}B + \bar{A}\gamma_\alpha D\bar{C}\gamma_\alpha B), \tag{2.21}$$

we conclude that after Fierzing $\delta L_1 + \delta L_3 = -\delta L_2$. This means that in the $\mathcal{N}=8$ case we have instead that $\delta L_1 + \delta L_3 = -\frac{1}{8}\delta L_2$ and we are missing $\frac{7}{8}\delta L_2$ which must come from Fierzing δL_4.

That this in fact is exactly what happens is most easily seen by Fierzing δL_4 keeping the factors $\gamma_\alpha\gamma_\beta f^\alpha$ intact and collecting the Γ^{ij} in the same factor. The result of the Fierzing is

$$(\bar{f}^\alpha\gamma_\beta\gamma_\alpha)\gamma^\mu\epsilon^{\nu\beta\delta}(\gamma_\gamma\gamma_\delta f^\gamma)\bar{\epsilon}\gamma_\mu\chi_\nu = 4\bar{f}^\mu(\gamma_\alpha\gamma_\beta f^\alpha)\left(\bar{\epsilon}\gamma^\beta\chi_\mu - \frac{1}{2}e_\mu{}^\beta\bar{\epsilon}\gamma^\sigma\chi_\sigma\right), \tag{2.22}$$

where the right hand side has been derived by writing $\gamma^{\nu\beta\delta}$ instead of $\epsilon^{\nu\beta\delta}$ and then multiplying in the explicit γ^μ into it.

Turning finally to the Fierz terms containing Γ_{ij} and Γ_{ijkl}, the latter terms can be seen to cancel directly using the same Fierz relations as for the terms without any Γ's. The cancelation of the Γ_{ij} does however require a separate calculation using the second basis set in the appendix. This cancelation has also been verified proving that the theory has $\mathcal{N}=8$ local supersymmetry.

We have also explicitly verified that the theory constructed here is locally scale invariant (denoted by an index Δ) and possesses $\mathcal{N}=8$ superconformal (shift) symmetry (denoted by S) with the following transformation rules (where ϕ is the local scale parameter and η the local shift parameter)

$$\begin{aligned}\delta_\Delta e_\mu{}^\alpha &= -\phi(x)e_\mu{}^\alpha,\\ \delta_\Delta\chi_\mu &= -\frac{1}{2}\phi(x)\chi_\mu,\\ \delta_\Delta B_\mu^{ij} &= 0,\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\delta_S e_\mu{}^\alpha &= 0,\\ \delta_S\chi_\mu &= \gamma_\mu\eta,\\ \delta_S B_\mu^{ij} &= \frac{i}{2}\bar{\eta}\Gamma^{ij}\chi_\mu.\end{aligned} \tag{2.24}$$

Verifying invariance under the latter transformations requires Fierz transformations similar to those used above to demonstrate $\mathcal{N} = 8$ supersymmetry. The calculations performed here may be facilitated by the use of the Mathematica package *GAMMA* [20].

3 The $\mathcal{N} = 8$ gauged BLG theory

In this section we first review the (ungauged) superconformal matter sector, i.e., the ordinary BLG theory, to which we then would like to couple the superconformal gravity derived in the previous section. The resulting "gauged" BLG theory is derived in the second subsection up to some higher order interaction terms between the two sectors.

3.1 Review of the ungauged $\mathcal{N} = 8$ superconformal BLG

The BLG theory contains three different fields; the two propagating ones $X^i{}_a$ and Ψ_a, which are three-dimensional scalars and spinors, respectively, and the auxiliary gauge field $\tilde{A}_\mu{}^a{}_b$. Here the indices a, b, ... are connected to the three-algebra and some n-dimensional basis T^a, while the i, j, k, ... indices are SO(8) vector indices. The spinors transform under a spinor representation of SO(8) but the corresponding index is not written out explicitly. Indices μ, ν, ... are vector indices on the flat M2-brane world volume.

Using these fields one can write down $\mathcal{N} = 8$ supersymmetry transformation rules and covariant field equations. This is possible without introducing a metric on the three-algebra. In such a situation the position of the indices on the structure constants is fixed as $f^{abc}{}_d$. The corresponding fundamental identity needed for supersymmetry and gauge invariance then reads [1–4],

$$f^{abc}{}_g f^{efg}{}_d = 3 f^{ef[a}{}_g f^{bc]g}{}_d \,, \tag{3.1}$$

which can be written in the following alternative but equivalent form [21],

$$f^{[abc}{}_g f^{e]fg}{}_d = 0 \,. \tag{3.2}$$

The construction of a Lagrangian requires the introduction of a metric on the three-algebra. As discussed above, if one wants to describe more general Lie algebras than $so(4)$, this metric must be degenerate [21] or non-degenerate but indefinite [22–24]. Finally, to construct an action one also needs to introduce the basic gauge field $A_{\mu ab}$ [4] which is related to the previously defined gauge field and structure constants as follows:

$$\tilde{A}_\mu{}^a{}_b = A_{\mu cd} f^{cda}{}_b \,. \tag{3.3}$$

The BLG Lagrangian is [3]

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(D_\mu X^{ia})(D^\mu X^i{}_a) + \frac{i}{2}\bar{\Psi}^a \gamma^\mu D_\mu \Psi_a - \frac{i}{4}\bar{\Psi}_b \Gamma_{ij} X^i{}_c X^j{}_d \Psi_a f^{abcd} \\ & - V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}\right) , \end{aligned} \tag{3.4}$$

[4]However, already gauge invariance of the field equations requires the introduction of this gauge field [21].

JHEP03(2009)074

where the potential is given by

$$V = \frac{1}{12} f^{abcd} f^{efg}{}_d X^i{}_a X^j{}_b X^k{}_c X^i{}_e X^j{}_f X^k{}_g \,. \tag{3.5}$$

Note that in terms of $\tilde{A}$ the Chern-Simons term becomes

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(A_{\mu ab}\partial_\nu \tilde{A}_\lambda{}^{ab} + \frac{2}{3} A_\mu{}^a{}_b \tilde{A}_\nu{}^b{}_c \tilde{A}_\lambda{}^c{}_a\right) \tag{3.6}$$

The BLG transformation rules for (global) $\mathcal{N} = 8$ supersymmetry are

$$\begin{aligned} \delta X^a_i &= i\epsilon\Gamma_i\Psi^a, \\ \delta\Psi_a &= \bar{D}_\mu X^i_a \gamma^\mu \Gamma^i \epsilon + \frac{1}{6} X^i_b X^j_c X^k_d \Gamma^{ijk}\epsilon f^{bcd}{}_a, \\ \delta\tilde{A}_\mu{}^a{}_b &= i\bar{\epsilon}\gamma_\mu\Gamma^i X^i_c \Psi_d f^{cda}{}_b. \end{aligned} \tag{3.7}$$

3.2 Coupling $\mathcal{N} = 8$ conformal supergravity to BLG matter

We now turn to the construction of the gauged BLG Lagrangian. The coupling of the BLG theory to the $\mathcal{N} = 8$ conformal supergravity theory discussed in the previous section follows from standard techniques. As a first step in its derivation we restrict ourselves to terms in the Lagrangian that give rise to $(cov.der.)^2$ or $(cov.der.)^3$ terms when varied under supersymmetry and show that all such terms cancel in δL. Including also some other terms, like those that complete the supercurrent, we will use the following Lagrangian as our starting point

$$L = L_{conf.sugra} + L^{cov}_{BLG} + L_{supercurrent}, \tag{3.8}$$

where $L_{conf.sugra}$ was given in section two,

$$L^{cov}_{BLG} = e\left(-\frac{1}{2}g^{\mu\nu}\tilde{D}_\mu X^{ia}\tilde{D}_\nu X_{ia} + \frac{i}{2}\bar{\Psi}^a\gamma^\alpha e_\alpha{}^\mu \tilde{D}_\mu\Psi_a + L_{Yukawa} - V\right) + L_{CS(A)}, \tag{3.9}$$

and

$$\begin{aligned} L_{supercurrent} = {} & Aie\bar{\chi}_\mu\Gamma^i\gamma^\nu\gamma^\mu\Psi^a\left(\tilde{D}_\nu X^{ia} - \hat{A}\frac{i}{2}\bar{\chi}_\nu\Gamma^i\Psi^a\right) \\ & + \hat{B}ie\bar{\chi}_\mu\gamma^\mu\Gamma^{ijk}\Psi_a(X^i_b X^j_c X^k_d) f^{abcd} + \hat{C}e\bar{\chi}_\mu\Gamma^{ijkl}\chi^\mu(X^i_a X^j_b X^k_c X^l_d) f^{abcd}, \end{aligned} \tag{3.10}$$

and then add terms as they become necessary for proving supersymmetry to the order in covariant derivatives at which we are working. Here the derivatives are covariant under all local symmetries of the theory. Note that the hatted parameters $\hat{A}$, $\hat{B}$ and $\hat{C}$ in the supercurrent are not determined by the $(\tilde{D}_\mu)^2$ calculation. In fact, at the end of this subsection we will determine also these coefficients by demanding cancelation of terms that contain fewer derivatives but are of power four or higher in X. The whole Lagrangian is then known up to some fermion terms without derivatives that might be needed in addition to the ones already present in the covariant derivatives. The final step of proving cancelation also of the one- and non-derivative terms in δL is fairly elaborate and will be presented elsewhere.

The new terms that arise in the computation are the following

$$A' i\epsilon^{\mu\nu\rho}\bar{\chi}_\mu \Gamma^{ij}\chi_\nu (X^i_a \tilde{D}_\rho X^j_a) + A'' i\bar{f}^\mu \Gamma^i \gamma_\mu \Psi_a X^i_a, \tag{3.11}$$

together with

$$-\frac{e}{16}X^2\tilde{R} + A'''\frac{i}{4}X^2\bar{f}^\mu\chi_\mu, \tag{3.12}$$

where the curvature term[5] is well-known to have exactly the coefficient $-\frac{1}{16}$ in three dimensions so that when added to the scalar kinetic term one obtains a locally scale invariant expression.

Recalling the way the transformation rules for the gauge fields A_μ and B_μ are obtained we infer that both δA^{ab}_μ and δB^{ij}_μ will pick up new terms in the process of constructing the coupled theory. This is natural in view of the fact that we work on-shell and that such terms are expected to arise when auxiliary fields are eliminated. We start from the following basic transformation rules without such terms

$$\begin{aligned}
\delta e_\mu{}^\alpha &= i\bar{\epsilon}_g\gamma^\alpha\chi_\mu, \\
\delta\chi_\mu &= \tilde{D}_\mu\epsilon_g, \\
\delta B^{ij}_\mu &= -\frac{i}{2}\bar{\epsilon}_g\Gamma^{ij}\gamma_\nu\gamma_\mu f^\nu, \\
\delta X^a_i &= i\epsilon_m\Gamma_i\Psi^a, \\
\delta\Psi_a &= (\tilde{D}_\mu X^i_a - i\hat{A}\bar{\chi}_\mu\Gamma^i\Psi_a)\gamma^\mu\Gamma^i\epsilon_m + \frac{1}{6}X^i_b X^j_c X^k_d \Gamma^{ijk}\epsilon_m f^{bcd}{}_a, \\
\delta\tilde{A}_\mu{}^a{}_b &= i\bar{\epsilon}_m\gamma_\mu\Gamma^i X^i_c\Psi_d f^{cda}{}_b,
\end{aligned} \tag{3.13}$$

where ϵ_g and ϵ_m are the supersymmetry parameters in the gravity and matter (BLG) sector, respectively. We are here using different supersymmetry parameters in the two sectors since, as we will see below, it will be necessary to rescale the supersymmetry parameters relative each other for the Lagrangian to be invariant.

As just mentioned, both $\delta\tilde{A}^{ab}_\mu$ and δB^{ij}_μ will pick up a number of new terms as we proceed with the calculation. By inspecting the possible terms we conclude directly that these additional pieces will not contain any derivatives $\tilde{D}_\mu$. Some of these are (with multiplicative constants and supersymmetry parameters to be determined)

$$\delta A^{ab}_\mu|_{new} = A_1\bar{\chi}_\mu\Gamma^{ij}\epsilon X^i_a X^j_b, \tag{3.14}$$

and

$$\delta B^{ij}_\mu|_{new} = B_1\bar{\Psi}^a\gamma_\mu\Gamma^{[i}\epsilon X^{j]}_a + B_2\bar{\chi}_\mu\Gamma^{k[i}\epsilon X^{j]}_a X^k_a + B_3\bar{\Psi}_a\Gamma^k\Gamma^{ij}\gamma_\mu\epsilon X^k_a. \tag{3.15}$$

[5] In the original version of this paper the curvature term was induced from a shift in the spin connection by $-\frac{1}{16}\epsilon_{\mu\alpha\beta}X^2$. This is correct to order $(cov.der)^2$ but is in general not compatible with BLG as the flat limit of the gauged theory.

JHEP03(2009)074

We may find still others as we go through the proof of supersymmetry at the $(\tilde{D}_\mu)^2$ level. It is important to note in this context that these new terms will not feed back into the proof of supersymmetry at the order which we are working here, namely $(\tilde{D}_\mu)^2$. The proof that the lower order terms in δL also cancel will, however, be affected.

The first step is to vary $L_{conf.sugra} + L^{cov}_{BLG}$ and keep only the $(\bar{D}_\mu)^2$ terms that are not directly canceled, along with all $(\bar{D}_\mu)^3$, in the pure supergravity case. That is, we here use the fact that the supergravity sector is invariant by itself as proved in the previous section. This means that we can drop the torsion part which is not a derivative term. When this is done it is possible to integrate by parts without problems. We find

$$\begin{aligned}\delta L_{conf.sugra}|_{D^2} + \delta L^{cov}_{BLG}|_{D^2} = & -\frac{1}{2}\delta(eg^{\mu\nu})D_\mu X^i_a D_\nu X^i_a - eD^\mu X^i_a D_\mu \delta X^i_a \\ & + ie\bar{\Psi}^a\gamma^\mu D_\mu \delta\Psi^a + e\delta B_{\mu ij}|_{grav}(X^i_a D^\mu X^j_a) \\ & + \frac{1}{2}\epsilon^{\mu\nu\rho}\delta A^{ab}_\mu|_{BLG+new}\tilde{F}^{ab}_{\nu\rho} + \epsilon^{\mu\nu\rho}\delta B^{ij}_\mu|_{new}G^{ij}_{\nu\rho}, \end{aligned} \tag{3.16}$$

where the fourth term on the right hand side contributes to $(D_\mu)^2$ only if we insert the original supergravity variation of δB^{ij}_μ as indicated. From now on we will not include the last two terms proportional to the field strengths $\tilde{F}^{ab}_{\mu\nu}$ and $G^{ij}_{\mu\nu}$ explicitly in our expressions. When the variations of the potentials need to be corrected we just have to recall their form from the above expression.

Computing the above variation gives

$$\begin{aligned}\delta L_{conf.sugra}|_{D^2} + \delta L^{cov}_{BLG}|_{D^2} = & -ieD_\mu X^i_a D_\nu X^i_a \left(\bar{\chi}^\mu\gamma^\nu\epsilon_g - \frac{1}{2}g^{\mu\nu}\bar{\chi}_\rho\gamma^\rho\epsilon_g\right) \\ & + \frac{i}{2}\epsilon^{\mu\nu\rho}\bar{\Psi}^a\gamma_\rho\Gamma^i\epsilon_m G_{\mu\nu ij}X^j_a + ie\bar{\Psi}^a\gamma^\mu\gamma^\nu\Gamma^i D_\mu\epsilon_m(D_\nu X^i_a) \\ & + \frac{i}{2}\bar{f}^\nu\gamma_\mu\gamma_\nu\Gamma^{ij}\epsilon_g(X^i_a D^\mu X^j_a), \end{aligned} \tag{3.17}$$

where we find the first new contribution to the variation of δB^{ij}_μ. Choosing $B_1 = -\frac{i}{2}$ and $\epsilon = \epsilon_m$ will then remove the $G_{\mu\nu}$ term from this expression.

To eliminate some of the other terms we now add the first part of the supercurrent $L_{supercurrent}|_{D^2} = Aie\bar{\chi}_\mu\Gamma^i\gamma^\nu\gamma^\mu\Psi^a\tilde{D}_\nu X^{ia}$ (the other terms do not contribute to $(D_\mu)^2$ when varied under supersymmetry). $(D_\mu)^2$ terms come from the variations $\delta\chi_\mu$ and $\delta\Psi_a$ which leave three terms (containing Γ^{ij}, Γ^i and no Γ^i's) two of which cancel the first and third terms on the right hand side above provided $A\epsilon_g = \epsilon_m$ and $2A\epsilon_m = \epsilon_g$. Thus we conclude that

$$\epsilon_m := \epsilon, \quad \epsilon_g = \pm\sqrt{2}\epsilon, \quad A = \pm\frac{1}{\sqrt{2}}, \tag{3.18}$$

where the sign of A will be chosen later.

The remaining terms are then

$$\delta L|_{D^2} = -Ai\epsilon^{\mu\nu\rho}\bar{\chi}_\mu\Gamma^{ij}\epsilon_m(D_\nu X^i_a D_\rho X^j_a) + \frac{i}{2}\bar{f}^\nu\gamma_\mu\gamma_\nu\Gamma^{ij}\epsilon_g(X^i_a D^\mu X^j_a). \tag{3.19}$$

We now add to L the term

$$A' i\epsilon^{\mu\nu\rho}\bar{\chi}_\mu\Gamma^{ij}\chi_\nu(X^i_a\tilde{D}_\rho X^j_a), \tag{3.20}$$

since when χ_μ is varied and the resulting expression integrated by parts the term above proportional to A is canceled if we choose $A' = -\frac{1}{4}$. We also find new contributions to the variations of $\delta\tilde{A}_\mu^{ab}$ and δB_μ^{ij} corresponding to $A_1 = 2iA', \epsilon = \epsilon_g$ and $B_2 = iA', \epsilon = \epsilon_g$.

Due to a second cancelation, arising for $A' = -\frac{1}{4}$ (where from now on we will choose the signs as $A = \frac{1}{\sqrt{2}}$, $\epsilon_g = \sqrt{2}\epsilon$), in the previous step only one term remains at this stage, namely

$$-\frac{i}{2}\bar{f}^\nu\gamma_\nu\gamma^\mu\Gamma^{ij}\epsilon_m(X_a^i D_\mu X_a^j). \tag{3.21}$$

Thus also the term

$$A'' i\bar{f}^\mu\gamma_\mu\Gamma^i\Psi_a X_a^i \tag{3.22}$$

is needed, where the variation of both χ_μ and Ψ_a will produce $(D_\mu)^2$ terms. All Γ^{ij} terms are eliminated by choosing $A'' = \frac{1}{\sqrt{2}}$ and $B_3 = -A''\frac{i}{16}, \epsilon = \epsilon_g$. This leaves us with the following variation (recalling that R^{**} is a double density)

$$\delta L|_{D^2} = -A''\frac{i}{4e}R^{**}\bar{\epsilon}_g\Gamma^i\Psi_a X_a^i + A''\frac{i}{2}(D_\mu X^2)\left(\bar{f}^\mu\epsilon_m - \frac{1}{e}\epsilon^{\mu\nu\rho}\bar{f}_\nu\gamma_\rho\epsilon_m\right). \tag{3.23}$$

Finally, we include the gravity term that is necessary to make the scalar field kinetic term locally scale invariant (which fixes the coefficient as given), that is,

$$L_R = -\frac{e}{16}\tilde{R}X^2. \tag{3.24}$$

We will also need the associated fermionic term

$$L_{ferm} = A''' iX^2\bar{f}^\mu\chi_\mu. \tag{3.25}$$

The variation of the Ricci scalar term reads

$$\delta L_R|_{D^2} = \frac{i}{4e}R^{**}X_a^i\bar{\epsilon}_m\Gamma^i\Psi_a - \frac{i}{8e}\bar{\chi}^\mu\gamma^\nu\epsilon_g R_{\mu\nu}^{**}X^2 + \frac{i}{4}\epsilon^{\mu\nu\rho}(D_\mu X^2)\bar{f}_\nu\gamma_\rho\epsilon_g. \tag{3.26}$$

Adding this to the last expression above for $\delta L|_{D^2}$ we see that the first terms in these two expressions cancel against each other, as do the last terms, provided we use the fact that $\epsilon_g = \sqrt{2}\epsilon_m$ as found above.

Thus, after including also the curvature scalar term we have

$$\delta L|_{D^2} = A''\frac{i}{2}(D_\mu X^2)\bar{f}^\mu\epsilon_m - \frac{i}{8e}\bar{\chi}^\mu\gamma^\nu\epsilon_g R_{\mu\nu}^{**}X^2. \tag{3.27}$$

The final step is then to add the variation of the fermionic term, that is,

$$\delta L_{ferm}|_{D^2} = A''' iX^2\bar{f}^\mu D_\mu\epsilon_g + A''' iX^2\bar{\chi}_\mu\delta f^\mu, \tag{3.28}$$

where the last term becomes, up to a $G_{\mu\nu}^{ij}$ field strength term,

$$A''' i\frac{1}{4e}\bar{\chi}^\mu\gamma^\nu\epsilon_g R_{\mu\nu}^{**}X^2. \tag{3.29}$$

JHEP03(2009)074

(This can also be expressed in terms of the ordinary Ricci tensor as

$$-A'''i\frac{e}{4}\left(\bar{\epsilon}_g\gamma_\mu\chi_\nu - \frac{1}{2}g_{\mu\nu}\bar{\epsilon}_g\gamma^\rho\chi_\rho\right)R^{\mu\nu}X^2, \tag{3.30}$$

where we used the relations

$$R^{**}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \quad R^{**} = -\frac{1}{2}R, \tag{3.31}$$

between the double dual $R^{**}_{\mu\nu}$ and the ordinary Ricci tensor.)

The $G^{ij}_{\mu\nu}$ term mentioned in the previous paragraph implies the following addition to δB^{ij}_μ:

$$\delta B^{ij}_\mu|_{new:S} = \frac{i}{64}X^2\bar{\epsilon}_g\Gamma^{ij}\chi_\mu, \tag{3.32}$$

which is just a special superconformal transformation with parameter

$$\eta = \frac{1}{32}X^2\epsilon_g. \tag{3.33}$$

This indicates that also $\delta\chi_\mu$ will pick up a special superconformal piece:

$$\delta\chi_\mu|_S = \gamma_\mu\eta = \frac{1}{32}X^2\gamma_\mu\epsilon_g = -\frac{1}{16\sqrt{2}}X^2\gamma_\mu\epsilon. \tag{3.34}$$

As we will see below this will, in fact, not happen. However, another contribution to δB^{ij}_μ will arise in the computation just below that will exactly double the above special superconformal part of this transformation rule.[6]

Summing up the situation at this point, using what we know about the various constants, we find that

$$\delta L|_{D^2} = \sqrt{2}A'''i\bar{f}^\mu D_\mu\epsilon X^2 + \frac{i}{2\sqrt{2}}\bar{f}^\mu\epsilon D_\mu X^2 - \frac{i}{4\sqrt{2}e}(1-2A''')\bar{\chi}^\mu\gamma^\nu\epsilon R^{**}_{\mu\nu}X^2. \tag{3.35}$$

Thus if we choose $A''' = \frac{1}{4}$ the first two terms add and the result can be integrated by parts to give

$$-\frac{i}{2\sqrt{2}}\bar{\epsilon}D_\mu f^\mu X^2 = \frac{i}{8\sqrt{2}e}\bar{\chi}^\mu\gamma^\nu\epsilon R^{**}_{\mu\nu}X^2, \tag{3.36}$$

modulo another $G^{ij}_{\mu\nu}$ term, and hence we see that the $(D_\mu)^2$ terms vanish in the variation of the Lagrangian.

We now turn to the hatted coefficients in the Lagrangian given in the beginning of this subsection. These can be determined as follows. Consider first $\hat{A}$. This parameter is fixed by demanding that the variation of Ψ_a is supercovariant,[7] which gives $\hat{A} = \frac{1}{\sqrt{2}}$.

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[6] We thank Xiaoyong Chu for pointing out a sign error in the first version of the paper.

[7] That is, if varied the right hand side of $\delta\Psi_a$ must not give rise to derivatives on the supersymmetry parameter.

JHEP03(2009)074

Turning to $\hat{B}$, we see that the variation of the dreibein in the covariantized sixth order potential term is canceled by choosing $\hat{B} = \frac{1}{6\sqrt{2}}$. The $\hat{B}$ term also gives rise to a X^6 term containing $\Gamma^{ijmn}(X^i_b X^j_c X^k_d) f^{abcd}(X^m_e X^n_f X^k_g) f^{aefg}$ which implies antisymmetry in $[bcef]$. However, this does not immediately mean that the fundamental identity will set it to zero, but by imposing $[abcf]$ on the fundamental identity (3.1) and using its alternative form given in (3.2), that sets the left hand side to zero, one finds that $f^{ab[cd}f^{ef]ag} = 0$ which is what we need.

The third, and last, parameter to be determined is $\hat{C}$. This we do by relating the $\delta\chi_\mu$ variation of this term to the two terms obtained by varying $\Psi_a|_{DX}$ in the $\hat{B}$ term and $\delta\Psi_a|_{X^3}$ in the supercurrent. We find $\hat{C} = 0$ due to a delicate cancelation.

We end this subsection by summarizing our results: Up to terms[8] of order three or higher in χ_μ, the Lagrangian reads

$$\begin{aligned}
L^{top}_{BLG} = &= \frac{1}{2}\epsilon^{\mu\nu\rho} Tr_\alpha \left(\tilde{\omega}_\mu \partial_\nu \tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu \tilde{\omega}_\nu \tilde{\omega}_\rho\right) - \epsilon^{\mu\nu\rho} Tr_i \left(B_\mu \partial_\nu B_\rho + \frac{2}{3} B_\mu B_\nu B_\rho\right) \\
&-ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu \bar{\chi}_\nu \gamma_\beta \gamma_\alpha \tilde{D}_\rho \chi_\sigma) \\
&+e\left(-\frac{1}{2}g^{\mu\nu}\tilde{D}_\mu X^{ia}\tilde{D}_\nu X_{ia} + \frac{i}{2}\bar{\Psi}^a \gamma^\alpha e_\alpha{}^\mu \tilde{D}_\mu \Psi_a + L_{Yukawa} - V\right) + L_{CS(A)} \\
&+\frac{1}{\sqrt{2}} ie\bar{\chi}_\mu \Gamma^i \gamma^\nu \gamma^\mu \Psi^a \left(\tilde{D}_\nu X^{ia} - \frac{i}{2\sqrt{2}}\bar{\chi}_\nu \Gamma^i \Psi^a\right) \\
&-\frac{1}{6\sqrt{2}} ie\bar{\chi}_\mu \gamma^\mu \Gamma^{ijk}\Psi_a (X^i_b X^j_c X^k_d) f^{abcd} \\
&-\frac{i}{4}\epsilon^{\mu\nu\rho}\bar{\chi}_\mu \Gamma^{ij}\chi_\nu (X^i_a \tilde{D}_\rho X^j_a) + \frac{i}{\sqrt{2}}\bar{f}^\mu \Gamma^i \gamma_\mu \Psi_a X^i_a \\
&-\frac{e}{16}X^2 \tilde{R} + \frac{i}{4}X^2 \bar{f}^\mu \chi_\mu \,,
\end{aligned} \tag{3.37}$$

and the transformation rules are

$$\begin{aligned}
\delta e_\mu{}^\alpha &= i\sqrt{2}\bar{\epsilon}\gamma^\alpha \chi_\mu \,, \\
\delta\chi_\mu &= \sqrt{2}\tilde{D}_\mu \epsilon, \\
\delta B^{ij}_\mu &= -\frac{i}{\sqrt{2}e}\bar{\epsilon}\Gamma^{ij}\gamma_\nu \gamma_\mu f^\nu - \frac{i}{2}\bar{\Psi}_a \gamma_\mu \Gamma^{[i}\epsilon X^{j]}_a - \frac{i}{2\sqrt{2}}\bar{\chi}_\mu \Gamma^{k[i}\epsilon X^{j]}_a X^k_a - \frac{i}{16}\bar{\Psi}_a \Gamma^k \Gamma^{ij}\gamma_\mu \epsilon X^k_a \\
&\quad + \frac{i}{16\sqrt{2}}\bar{\epsilon}\Gamma^{ij}\chi_\mu, \\
\delta X^a_i &= i\bar{\epsilon}\Gamma_i \Psi^a, \\
\delta\Psi_a &= (\tilde{D}_\mu X^i_a - \frac{i}{\sqrt{2}}\bar{\chi}_\mu \Gamma^i \Psi_a)\gamma^\mu \Gamma^i \epsilon + \frac{1}{6}X^i_b X^j_c X^k_d \Gamma^{ijk}\epsilon f^{bcd}{}_a, \\
\delta\tilde{A}_\mu{}^a{}_b &= i\bar{\epsilon}\gamma_\mu \Gamma^i X^i_c \Psi_d f^{cda}{}_b - \frac{i}{\sqrt{2}}\bar{\chi}_\mu \Gamma^{ij}\epsilon X^i_c X^j_d f^{cda}{}_b.
\end{aligned} \tag{3.38}$$

What remains to be checked are the terms in δL that are linear in the covariant derivative or independent of them. We hope to present this final step of the proof elsewhere.

[8]It might be that all terms of this kind are already accounted for by the ones built into the covariant derivative in which case the presented Lagrangian is the complete answer. Note that terms like $\bar{\chi}_\mu \Gamma^{ijkl}\chi^\mu \bar{\Psi}_a \Gamma^{ijkl}\Psi^a = 0$ due to the chirality properties, and that $\bar{\chi}_\mu \chi^\mu \bar{\Psi}_a \Psi^a$ is already present in the supercurrent.

JHEP03(2009)074

4 Conclusions and comments

In this paper we have constructed the $\mathcal{N}=8$ conformal supergravity theory in three dimensions that seems to be the proper theory to couple to the $\mathcal{N}=8$ BLG theory believed to describe two M2 branes at the IR conformal fix-point. The $\mathcal{N}=8$ conformal supergravity theory consists on-shell of just three Chern-Simons type terms one for each of the gauge fields, the spin connection (in second order form), the Rarita-Schwinger and SO(8) R-symmetry gauge fields. This theory should be possible to couple to matter in the form of the BLG theory which is a rather lengthy operation to do in full detail. The construction carried out in this paper, relying on the cancelation in δL of terms containing two or three covariant derivatives, generates the complete Lagrangian apart from some fermionic interaction terms.

There are several aspects of the gauged BLG theory that might be of interest. Free Chern-Simons gauge theories are really topological theories whose symmetries become reduced to superconformal ones when coupled to each other (as in the supergravity sector) or to conformal matter (as in the BLG sector) although the gravity sector is probably somewhat more intricate. In any case, what seems to be a general feature is that the various curvatures are heavily restricted, or even determined, by the field equations. For pure Chern-Simons gravity this is discussed for instance in [15]. A Lagrangian based on a second order spin connection leads to the equation of motion

$$D_{[\mu}W_{\nu]\rho}=0,\ \ W_{\mu\nu}=R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R, \tag{4.1}$$

which is known to be the condition for conformal flatness in three dimensions. This equation will be modified by source terms constructed from the other fields appearing in the theory.

Another well-known property of the BLG theory is that it allows for the introduction of a level k [8, 9, 25] which can be seen by using structure constants of the form

$$f^{abcd}=\frac{2\pi}{k}\epsilon^{abcd}. \tag{4.2}$$

Then reabsorbing one such factor $\frac{2\pi}{k}$ into the gauge field in the BLG theory produces the level k theory where k is an integer for topological reasons. If this is done in the Van Raamsdonk version [25] one finds the standard level $(k,-k)$ theory discussed more generally in [10]. Interestingly enough the coupling to superconformal gravity discussed in this paper introduces yet another level parameter which is also quantized as explained in [15]. There seems, however, to be room for only one new such parameter in the $\mathcal{N}=8$ superconformal case since the extra Chern-Simons terms are connected by the various local symmetries. It is perhaps interesting to note in this context that the Chern-Simons term for the R-symmetry field B_μ^{ij} gets here an unconventional normalization (being twice the standard one).

This last issue relates also to the question of invariance under parity for the gauged BLG theory. The pure BLG theory is saved by the fact that in the SU(2) $\times$ SU(2) formulation in [8, 9, 25] the two gauge groups are interchanged by a parity transformation. This option seems not to be available in the superconformal gravity sector as formulated here.

Apart from the original $\mathcal{N}=8$ BLG theory there are a number of other versions of superconformal M2 brane theories with less supersymmetry but able to describe more general stacks of branes. Following [25], the authors of [10] (see also [26, 27]) used a construction with fields in the bi-fundamental representation of $\mathrm{U}(N)\times\mathrm{U}(N)$ relevant for stacks with N branes. So far, however, this ABJM theory exists only with 6 supersymmetries which, however, may get enhanced to 8 for level $k=1,2$ if monopole operators are introduced [10, 28]. In such a context infinite dimensional algebraic structures will probably play a role. An example of such a structure, related to generalized Jordan triple products, has recently been suggested to arise in BLG/ABJM theories [29]. Here we have not made an attempt to couple the ABJM theory to $\mathcal{N}=6$ superconformal gravity but it should be possible and follow the same lines as those used in this paper. Another method that might be useful in this context is the embedding tensor technique already applied to similar problems, for instance, in [30].

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A Fierz bases

The Fierz basis used in the proof of supersymmetry in the main text is based on expressions of the form $(\bar{\epsilon}\ldots\chi_\mu)(\bar{f}_\nu\ldots f_\rho)$ where the dots refers to either an antisymmetric three-dimensional charge conjugation matrix or to a product of it with a three-dimensional gamma which is symmetric. Thus these expressions have three, four or five free indices that need to be contracted by deltas or Levi-Civita symbols. There are thus twelve index structures:

$$\begin{aligned}
&(-)\ \ (\bar{\epsilon}\chi_\mu)(\bar{f}_\nu f_\rho)\epsilon^{\mu\nu\rho}=0,\\
&(-)\ \ (\bar{\epsilon}\chi_\alpha)(\bar{f}^\beta\gamma^\alpha f_\beta)=0,\\
&(1)\ \ (\bar{\epsilon}\chi_\alpha)(\bar{f}^\alpha\gamma^\beta f_\beta),\\
&(2)\ \ (\bar{\epsilon}\gamma^\alpha\chi_\alpha)(\bar{f}^\beta f_\beta),\\
&(3)\ \ (\bar{\epsilon}\gamma_\alpha\chi_\beta)(\bar{f}^\alpha f^\beta),\\
&(4)\ \ (\bar{\epsilon}\gamma^\alpha\chi_\alpha)(\bar{f}_\mu\gamma_\nu f_\rho)\epsilon^{\mu\nu\rho},\\
&(5)\ \ (\bar{\epsilon}\gamma_\mu\chi_\nu)(\bar{f}_\rho\gamma^\alpha f_\alpha)\epsilon^{\mu\nu\rho},\\
&(6)\ \ (\bar{\epsilon}\gamma^\alpha\chi_\mu)(\bar{f}_\alpha\gamma_\nu f_\rho)\epsilon^{\mu\nu\rho},\\
&(7)\ \ (\bar{\epsilon}\gamma^\alpha\chi_\mu)(\bar{f}_\nu\gamma_\alpha f_\rho)\epsilon^{\mu\nu\rho},\\
&(8)\ \ (\bar{\epsilon}\gamma^\mu\chi_\alpha)(\bar{f}^\alpha\gamma_\nu f_\rho)\epsilon^{\mu\nu\rho},\\
&(9)\ \ (\bar{\epsilon}\gamma_\mu\chi_\alpha)(\bar{f}_\nu\gamma^\alpha f_\rho)\epsilon^{\mu\nu\rho},\\
&(-)\ \ (\bar{\epsilon}\gamma_\mu\chi_\nu)(\bar{f}_\alpha\gamma^\rho f_\alpha)\epsilon^{\mu\nu\rho}=0.
\end{aligned}\tag{A.1}$$

Of the ten non-zero ones we can easily (by cycling the three indices on the epsilon tensor together with on of the contracted indices) find three relations involving the expressions (4) to (9):

$$2 \cdot (6) = (4) + (9), \;\; 2 \cdot (5) = (7) - (9), \;\; (4) = 2 \cdot (6) - (7). \tag{A.2}$$

We will choose as an independent set of expressions (1), (2), (3), (4), (6), and (8), which means that $(9) = 2 \cdot (8) - (4)$, $(7) = 2 \cdot (6) - (4)$, and $(5) = (6) - (8)$.

We may also relate this basis to expressions that appear frequently in the Lagrangian:

$$\begin{aligned} (\hat{4}) &:= (\bar{\epsilon}\gamma^\alpha \chi_\alpha)(\bar{f}_\beta \gamma^\gamma \gamma^\beta f_\gamma) \\ (\hat{6}) &:= (\bar{\epsilon}\gamma^\alpha \chi_\beta)(\bar{f}_\alpha \gamma^\gamma \gamma^\beta f_\gamma), \\ (\hat{8}) &:= (\bar{\epsilon}\gamma^\alpha \chi_\beta)(\bar{f}_\beta \gamma^\gamma \gamma^\alpha f_\gamma). \end{aligned} \tag{A.3}$$

Expressing these in the basis specified above gives

$$(\hat{4}) = (4) + (2), \;\; (\hat{6}) = (6) + (3), \;\; (\hat{8}) = (8) + (3). \tag{A.4}$$

When the SO(8) Γ-matrices are introduced into the Fierz identity the same basis can be used by inserting Γ's into both factors. For Γ^{ijkl} the basis is exactly the same as the one above while for Γ^{ij} some other elements are set to zero by symmetry

$$\begin{aligned} &(1')\ (\bar{\epsilon}\Gamma^{ij}\chi_\mu)(\bar{f}_\nu \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho}, \\ &(2')\ (\bar{\epsilon}\Gamma^{ij}\chi_\alpha)(\bar{f}^\beta \gamma^\alpha \Gamma^{ij} f_\beta), \\ &(3')\ (\bar{\epsilon}\Gamma^{ij}\chi_\alpha)(\bar{f}^\alpha \gamma^\beta \Gamma^{ij} f_\beta), \\ &(-)\ (\bar{\epsilon}\Gamma^{ij}\gamma^\alpha\chi_\alpha)(\bar{f}^\beta f_\beta) = 0, \\ &(4')\ (\bar{\epsilon}\gamma_\alpha\Gamma^{ij}\chi_\beta)(\bar{f}^\alpha \Gamma^{ij} f^\beta), \\ &(-)\ (\bar{\epsilon}\gamma^\alpha\Gamma^{ij}\chi_\alpha)(\bar{f}_\mu \gamma_\nu \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho} = 0, \\ &(5')\ (\bar{\epsilon}\gamma_\mu\Gamma^{ij}\chi_\nu)(\bar{f}_\rho \gamma^\alpha \Gamma^{ij} f_\alpha)\epsilon^{\mu\nu\rho}, \\ &(6')\ (\bar{\epsilon}\gamma^\alpha\Gamma^{ij}\chi_\mu)(\bar{f}_\alpha \gamma_\nu \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho}, \\ &(-)\ (\bar{\epsilon}\gamma^\alpha\Gamma^{ij}\chi_\mu)(\bar{f}_\nu \gamma_\alpha \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho} = 0, \\ &(7')\ (\bar{\epsilon}\gamma^\mu\Gamma^{ij}\chi_\alpha)(\bar{f}^\alpha \gamma_\nu \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho}, \\ &(-)\ (\bar{\epsilon}\gamma_\mu\Gamma^{ij}\chi_\alpha)(\bar{f}_\nu \gamma^\alpha \Gamma^{ij} f_\rho)\epsilon^{\mu\nu\rho} = 0, \\ &(8')\ (\bar{\epsilon}\gamma_\mu\Gamma^{ij}\chi_\nu)(\bar{f}_\alpha \gamma^\rho \Gamma^{ij} f_\alpha)\epsilon^{\mu\nu\rho}, \end{aligned} \tag{A.5}$$

and the set of independent basis elements can be chosen as $(1')$, $(2')$, $(3')$, $(4')$, $(5')$ and $(7')$.

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JHEP03(2009)074

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